Quiz 3 solutions

1. Let G be a group, and let $H, N < G$ be solvable subgroups such that $N \lhd G$. Then show that internal product HN is solvable.

   Solution. From Lesson Plan 3.2 (iv), we know that $HN \leq G$, $N \trianglelefteq HN$, and $H \cap N \trianglelefteq H$. Thus, by the Second Isomorphism Theorem, it follows that $H/H \cap N \cong HN/N$. Furthermore, since H is solvable and $H \cap N \leq H$, it follows from Lesson Plan 7.2 (vii) that $H \cap N$ is solvable, Thus, it follows from Assignment (iv): Practice problem 9 that $H/H \cap N$ is solvable, and consequently HN/N is solvable. Finally, since N is solvable and $N \trianglelefteq HN$, by Lesson Plan 7.2 (viii), we conclude that HN is solvable.

2. Show that a group of order p^2q, where p and q are distinct primes with $p < q$, is solvable. [Hint: Use the Sylow's theorems and note that $p \not\equiv 1 \pmod{q}$.]

   Solution. Let G be of order p^2q. From the Third Sylow Theorem, we know that the number of Sylow q-subgroups (of G) $n_q \equiv 1 \pmod{q}$ and $n_q \mid p^2$. Thus, it follows that $n_q \in \{1, p, p^2\}$. Since $p < q$, we have that $p \not\equiv 1 \pmod{q}$, and hence $n_q \neq p$.

   Suppose that $n_q = p^2$. Then $p^2 \equiv 1 \pmod{q}$, or $q \mid p^2 - 1$, which implies that $q \mid (p-1)(p+1)$. Since $q > p$, it follows that $q = p + 1$, which forces that $p = 2$, $q = 3$, and $|G| = 12$. Since every abelian group is solvable, and the only non-abelian groups of order 12 (up to isomorphism) are A_4, D_{12}, and $\mathbb{Z}_4 \ltimes_{-1} \mathbb{Z}_3$, which are all solvable (use Assignment (iv): Practice problem 6 and Lesson Plan 3.2 (vii) to verify this!), we have that G is solvable.

   Finally, if $n_q = 1$, then G has a unique Sylow q-subgroup Q, which is normal in G, by Lesson Plan 4.4 (x). Furthermore, since $|G/Q| = p^2$, it follows that G/Q is abelian. Therefore, as Q is solvable, by Lesson Plan 3.2 (viii), we infer that G is abelian.